Computer Graphics

02. Rotations in 3D

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Ideal Rotation Format

- Represent degrees of freedom with minimum number of values
- Math should be simple and efficient for:
  - Concatenation
  - Interpolation

Topics

- Angle (2D) & Matrix (2D)
- Euler Angles (3D)
- Axis-Angle (3D) & Matrix (3D)

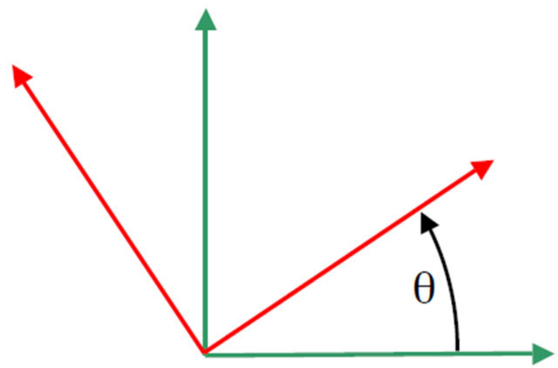
- Complex number (2D)
- Quaternion (3D)

Angle 2D. Rotation

Can we express this as a matrix?

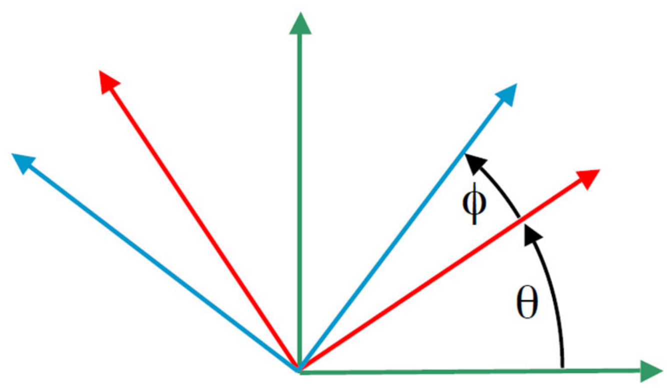
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R(\theta) = ?$$



Angle 2D. Composition

- Is linear
- Is commutative



Angle 2D. Interpolation

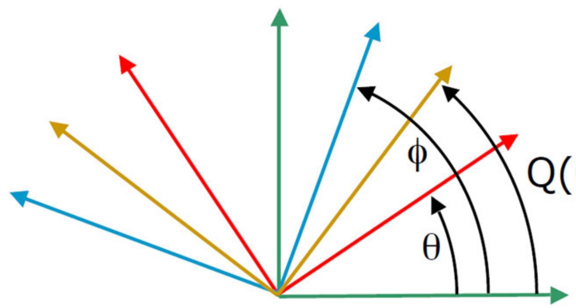
- Is linear
- Is commutative

- It works for:

$$Q = (1 - t)\theta + t\varphi$$

- Problems! what if:

$$\theta = 30^o$$
$$\varphi = 390^o$$



Angle 2D. Rotation

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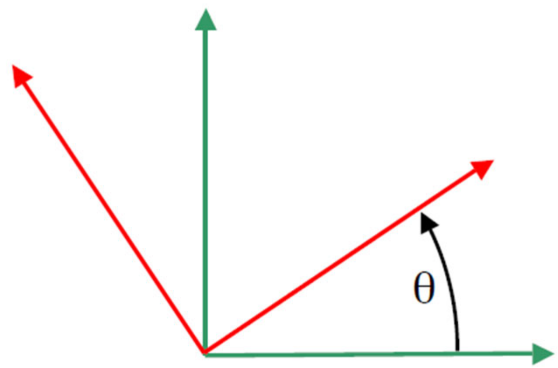
$$R(\theta) = ?$$



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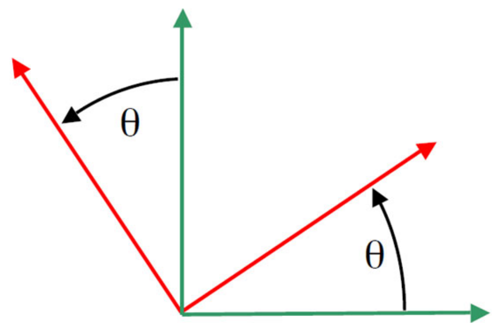
Angle 2D. Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R(\theta) = ?$$



Rotate the entire frame of reference!

Angle (2D) & Matrix (2D). Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

How to transform a rotation represented with an angle to a rotation represented with a matrix:

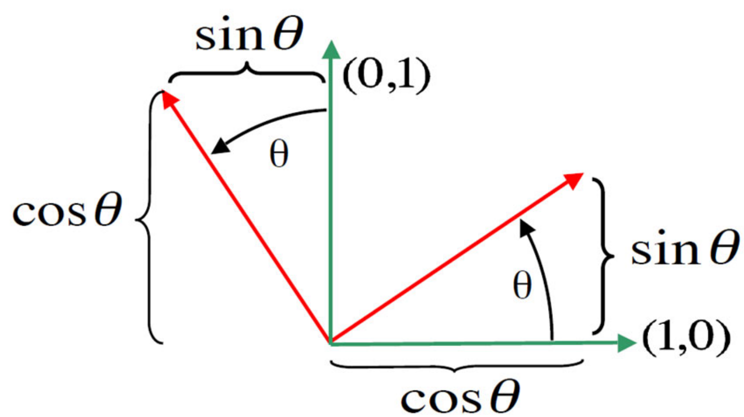
$$R(\theta) = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

How to build a Proof:

Consider

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, R(\theta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

With [1,0] find a and b from the geometric construct below. Then do the same for c and d



Angle (2D) & Matrix (2D). Summary

- Compact (in angle space 1 value)
- Rotation not ideal (in matrix space)

  (4 values for 1 degree of freedom)

  $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
- Concatenation easy

  (add in angle space)

  (product in matrix space)
- Interpolation doable

  (weighted sum in angle space)

  (unclear in matrix space)

Angle (3D). Euler Angles

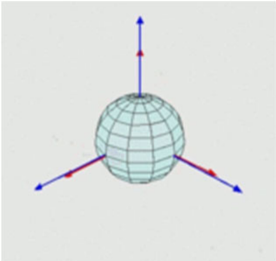
Rotation around 3 different axis, in FIXED order

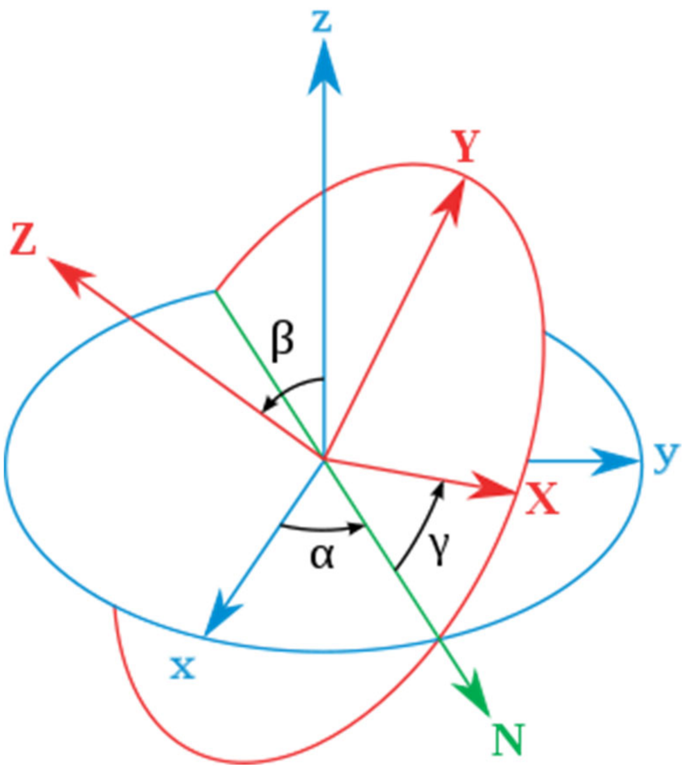
<u>Rotations in 3D are NOT commutative!</u>

- Extrinsic rotations (fixed axis)
- Intrinsic rotations (moving axis)
- Many Intrinsic options,

  for example: z-x’-z’’

Line of Node is intersection of planes





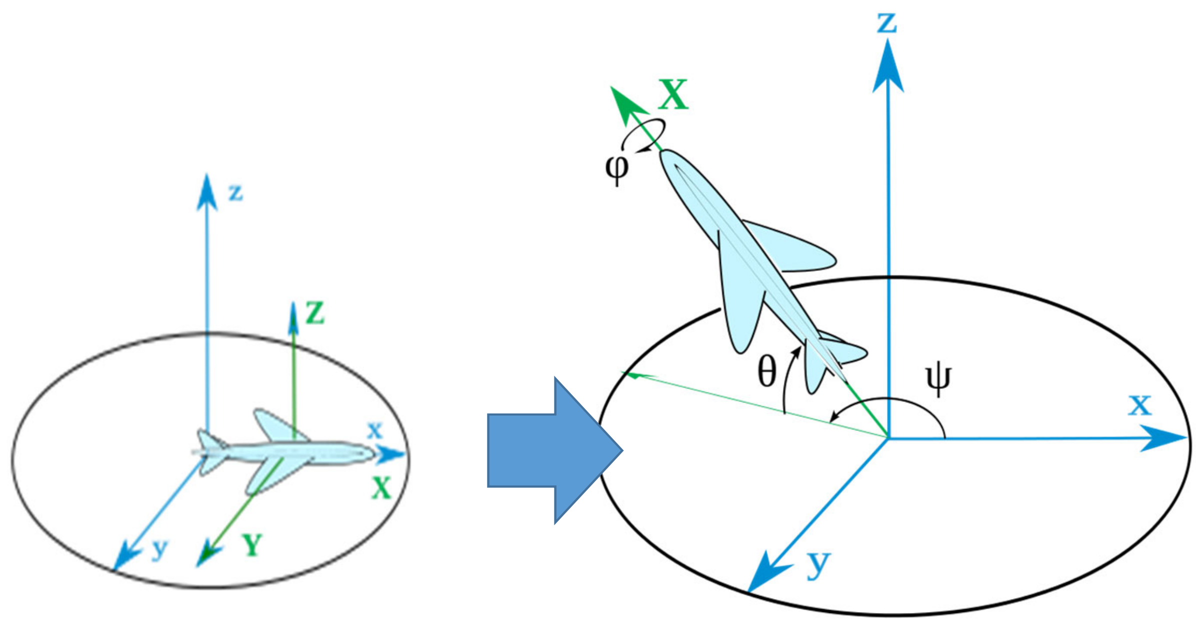
Angle (3D). Tait-Bryan rotations

Rotation around 3 different axis, in FIXED order

Example 2:

Tait-Bryan chained rotations:

Heading, elevation and bank angles after yaw, pitch and roll rotations (Z-Y'-X'')



Angle (3D). Trouble

- Gimbal Lock
- Different rotations give same angles. Example, on Euler angles:

$$(90,90,90) = (0,90,0)$$

Rotation on z cancels rotation on z''

(y rotation aligns them)

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Angle (3D). Summary

- Compact (in angle space 3 values)
- Rotation not ideal
- Concatenation not easy

  (angles processed sequentially)
- Interpolation unclear

- Additional trouble

  (gimbal lock)

  (same rotation with different values)

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Axis-Angle (3D)

1. It can be shown that

<u>any rotation in 3D can be done given one single rotation on a certain axis</u>

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2. We know that any vector can be expressed as a linear combination of three orthogonal vectors (x,y,z)

3. Therefore, we can make a linear combination if we express the axis as a vector

Axis-Angle (3D)

For a given angle θ and vector u, R can be expressed as:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z\left(1-\cos\theta\right) + u_y\sin\theta \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z\left(1-\cos\theta\right) - u_x\sin\theta \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \cos\theta + u_z^2\left(1-\cos\theta\right) \end{bmatrix}$$

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Axis-Angle (3D)

For a given angle θ and vector u , R can be expressed:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

```
glm::vec3 myRotationAxis( ??, ??, ??);
glm::rotate( angle_in_degrees, myRotationAxis );
```

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Axis-Angle (3D). Summary

- Compact (1 angle+ 3 axis coord.)
- Rotation robust but expensive

  (axis processed separately)

  (calculus on 9 values with complicated formula)
- Concatenation easy
- Interpolation unclear

- No additional trouble

  (NO gimbal lock)

  (unique representation)

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Reading and Review

https://www.essentialmath.com/GDC2012/GDC2012_JMV_Rotations.pdf

https://en.wikipedia.org/wiki/Rotation_matrix
https://en.wikipedia.org/wiki/Euler_angles
https://en.wikipedia.org/wiki/Davenport_chained_rotations

http://www.opengl-tutorial.org/beginners-tutorials/tutorial-3-matrices/#cumulating-transformations

Bourg et al. chapter 11

Summary:

- Rotations in 2D
  - Angle
  - Matrix
- Rotations in 3D
  - Euler Angles
  - Yaw-Pitch-Roll
  - Axis Angle
  - 3x3 Matrix

Rotations

- With complex numbers
- With quaternions

We want to have:

- Compact representation
- Simple calculation
- Robust composition
- Robust interpolation

Reminder on Complex Numbers

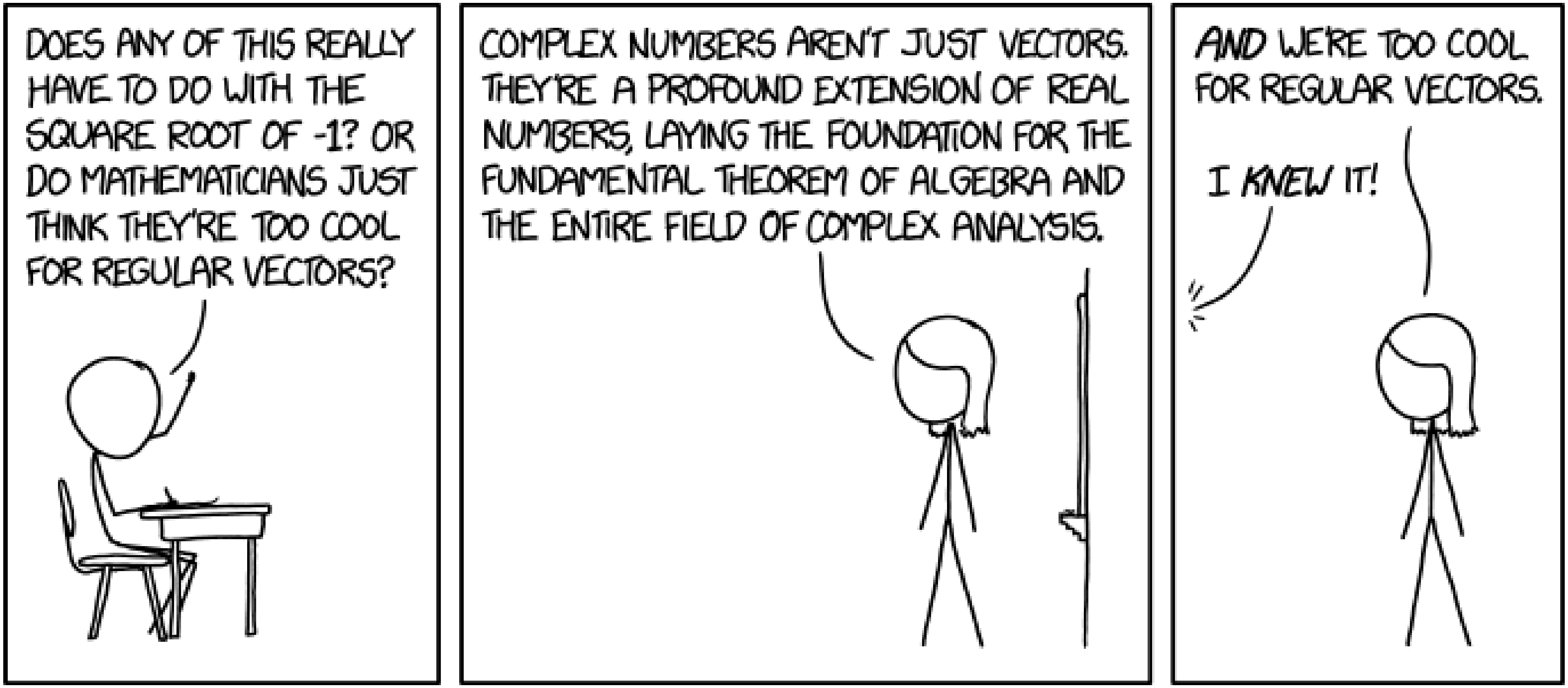
Definition:

$$z = a + bi$$

with

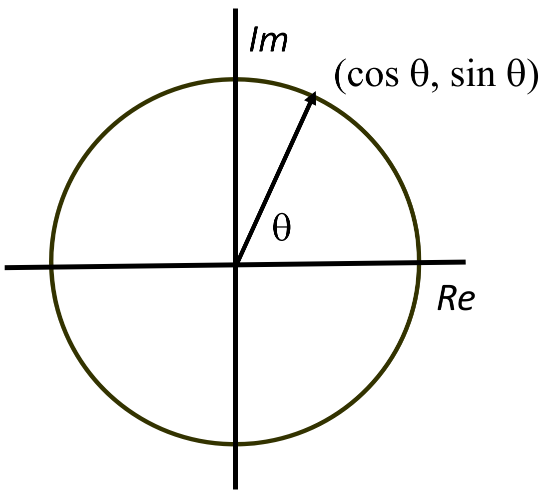
$$i^2 = -1$$

- Complex numbers are a good compact representation of movements on a plane (2 Values)
- If normalized $(a^2 + b^2 = 1)$, can use these to represent 2D rotation



https://xkcd.com/2028/

Unit circle on complex plane



Euler Formula
(proof from Taylor Expansions)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Polar form of a complex number:

$$a + bi = Ae^{i\theta}$$

Euler Identity

You may have seen this:

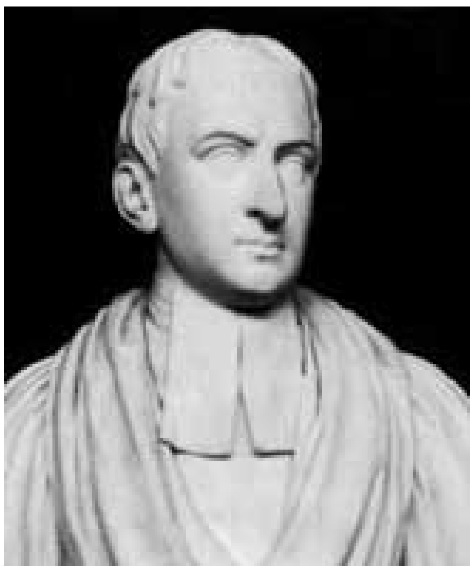
$$e^{\pi\mathbf{i}} + 1 = 0$$

It falls out from:

$$\begin{aligned} 0 &= e^{\pi\mathbf{i}} + 1 \\ &= \cos\pi + \mathbf{i}\sin\pi + 1 \\ &= -1 + \mathbf{i}(0) + 1 \\ &= 0 \end{aligned}$$

Who came up with this?

Roger Cotes in 1714 (sculpture by Scheemakers)



Euler in 1748 (painting by Handmann)

Interpretation on Plane

- Caspar Wessel (1799)
- Jean Robert Argand (1806)
- Made "popular" around 1814

Operations

Conjugate:
(a+bi)* = a -bi

Addition:
(a+bi)+(c+di) = (a+c)+ (b+d)i

Product:
(a+bi) (c+di)= (ac - bd) + (ad +bc)i

$$Ae^{i\theta} * Be^{i\varphi} = ABe^{i(\theta+\varphi)}$$

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Conclusion

A multiplication by a complex number of modulus 1 can be seen, in geometrical terms, as a rotation on the plane

Next class

Can we achieve simple and efficient maths for all three?

- Concatenation
- Interpolation
- Rotation

Next week:

- Complex numbers

  (good rotations in 2D)
- Quaternions

  (good rotations in 3D)

And how to go from there to usual space

In addition:

- Review on dot and cross product

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